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Black Hole Entropy: Membrane Approach

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The "wall contribution" character of the Bekenstein–Hawking entropy in the brick-wall model leads us to propose a new method of computing the entropy of a black hole. In our model, the entropy is attributed to the dynamical degrees of the field covering the two dimensional membranes just outside the horizon. A cutoff different from the model of 't Hooft is necessarily introduced. It can be treated as an increase in horizon because of the space–time fluctuations. It is also shown that our method can be applied to the nonstatic case, such as Vaidya–deSitter space–time, and the final result relies on a time-dependent cutoff different from the brick-wall model.

1. INTRODUCTION

One of the important problems in theoretical physics is the investigation of the statistical origin of black hole entropy. Although it is generally believed that the final answer depends upon how perfect the theory of quantum gravity is; in many works authors have used the semiclassical approach, such as the brick-wall model ('t Hooft, 1985), and the entanglement entropy interpretation (Bombelli *et al.*, 1986; Frolov and Novikov, 1993). The latter is associated with the unobservable modes hidden in the horizon. Our understanding of this interpretation is that the entropy of the black hole can be attributed to the decoherence of the quantum fields in the black hole background. It is shown that the entanglement interpretation is closely related to the brick-wall model (Kabat and Strassler, 1994). If a black hole originates from a star that is supposed to be in a pure state, there exists a perfect relation between the internal and external modes. In the brick-wall model proposed by 't Hooft, the statistical property of external field outside the hole is investigated in the brick-wall condition: The field is supposed to vanish near horizon and at large distances. The result includes three parts: The first term is proportional to the area of the horizon and depends on a cutoff η that stands for the proper distance

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between the wall and the horizon; the second term is the quantum correction to black hole entropy with a logarithmically divergent term; the third term has the same equation of the state of thermal radiation in flat space–time, this part is of little relevance to black hole entropy. It is also shown by Rindler approximation (Frolov and Fursaev, 1998; Li and Zhao, 2000) that the well-known Bekenstein–Hawking(B–H) entropy proportional to the area of the horizon is only derived from the contributions of the fields near the horizon. In other words, the B–H entropy comes from the "wall contribution" (Mukohyama and Israel, 1998). It is easily understood because the density of states becomes infinity at the horizon. Now let us recall it through a static black hole as follows:

$$ds^2 = -f \ dt^2 + f^{-1} \ dr^2 + r^2 \ d\Omega^2, \tag{1.1}$$

and the entropy of a scalar field in this background is given by (Jing, 1998)

$$S = \frac{2\pi^2}{45\beta^3} \int d\theta \ d\varphi \int_{r_0}^{L} \frac{\sqrt{-g}}{g_{tt}^2} \ dr, \tag{1.2}$$

where the relation $g_{tt}(r_0) = 0$ gives location the of horizon. In the original form of the brick-wall model, the upper limit of the integral satisfies the inequality $L \gg r_0$. This equation can give the quantum correction to black hole entropy besides the Bekenstein–Hawking entropy. However, the B–H entropy is only derived from the "wall contribution." In other words, if we want to get only the part proportional to the area of the horizon, the above integral can be done in the vicinity of the horizon and the upper limit of integral may be replaced by another quantity. A small quantity h is introduced and L is replaced by $r_0 + \epsilon + h$ in Eq. (1.2)

$$\begin{aligned} S &\simeq \frac{8\pi^3}{45\beta^3} \int_{r_0+\epsilon}^{r_0+\epsilon+h} \frac{r^2 \ dr}{f^2} \\ &= \frac{8\pi^3}{45\beta^3} \int_{r_0+\epsilon}^{r_0+\epsilon+h} \frac{r^2 \ dr}{4\kappa^2 (r - r_0)^2}, \end{aligned} \tag{1.3}$$

where $\epsilon \ll h \ll r_0$. The function $f(r)$ is expanded with Taylor series in the vicinity of the horizon

$$f(r) = f'(r_0)(r - r_0) = 2\kappa(r - r_0), \tag{1.4}$$

where $\kappa = \frac{1}{2} f'(r_0)$ is the surface gravity near the horizon. The leading contribution of (1.3) reads

$$S = \frac{2\pi^3 r_0^2}{45\beta^3 \kappa^2 \epsilon'}, \tag{1.5}$$

where $1/\epsilon' = h/\epsilon(\epsilon + h)$; the Hawking temperature $\beta^{-1} = \kappa/2\pi$. We redefine the cutoff as

$$\eta = \int_{r_0}^{r_0+\epsilon'} \frac{dr}{\sqrt{2\kappa(r - r_0)}} = \sqrt{\frac{2\epsilon'}{\kappa}}. \tag{1.6}$$

So we obtain

$$S = \frac{A}{360\pi\eta^2}. \tag{1.7}$$

If $\eta^2 = \frac{1}{90\pi}$, one gets the B–H entropy, which is a quarter of the area of the horizon.

What we have just investigated is a three-dimensional (3-D) system, not a two-dimensional (2-D) one. However, the "wall contribution" character of the entropy is so strongly impressed on us that an idea arises naturally: Perhaps we can compute the B–H entropy by investigating the statistical mechanics of the 2-D surface of event horizon as a membrane in the 4-D space–time. In fact, the Euclidean path integral method shows that the B–H entropy is derived from the surface term of the gravitational action only. It is woeful that by this method we know nothing but the thermodynamical entropy, and its statistical origin is unclear. It seems natural even that we calculate the black hole entropy by investigating the statistical mechanics of a 2-D membrane. In our opinion, the entropy of a black hole is attributable to only the degrees of freedom of the fields covering the S^2 surface of the event horizon, derived neither from the inner fields nor from the outward radiation fields.

2. MODEL IN A STATIC HOLE

A spherical surface just outside the horizon is fixed at $R = r_0 + \varepsilon$; ε is a small quantity and $dR = 0$. The geometry of the surface is described by

$$ds_3^2 = -f(R)\ dt^2 + R^2\ d\Omega^2, \tag{2.1}$$

and

$$\begin{aligned} \sqrt{-g} &= \sqrt{f}R^2\ \sin\theta, \quad g^{00} = -f^{-1}, \\ g^{11} &= \frac{1}{R^2}, \quad g^{22} = \frac{1}{R^2\ \sin^2\theta}. \end{aligned} \tag{2.2}$$

Substituting (2.2) into the following equation of a massless scalar field,

$$\frac{1}{\sqrt{-g}}\ \partial_\mu(\sqrt{-g}g^{\mu\nu}\ \partial_\nu\Phi) = 0, \tag{2.3}$$

We obtain

$$\frac{R^2}{f}\ \partial_t^2\Phi + \partial_\theta^2\Phi + \cot\theta\ \partial_\theta\Phi + \frac{1}{\sin^2\theta}\ \partial_\varphi^2\Phi = 0. \tag{2.4}$$

By using the WKB approximation with

$$\Phi = e^{-i\omega t + iS(\theta,\varphi)}, \tag{2.5}$$

we can define $p_\theta = \frac{\partial S}{\partial \theta}, \quad p_\varphi = \frac{\partial S}{\partial \varphi}$. Substituting (2.5) into (2.4), we have

$$\frac{\omega^2 R^2}{f} - p_\theta^2 - \frac{1}{\sin^2\theta} p_\varphi^2 = 0, \tag{2.6}$$

or

$$p_\varphi = \pm \sin\theta \left(\frac{\omega^2 R^2}{f} - p_\theta^2 \right)^{1/2}. \tag{2.7}$$

Therefore, in phase space we obtain the number of modes

$$\begin{aligned} g(\omega) &= \frac{1}{4\pi^2} \int d\theta \ d\varphi \int dp_\theta \ dp_\varphi \\ &= \frac{1}{4\pi^2} \int d\theta \ d\varphi \int dp_\theta \ \sin\theta \left(\frac{\omega^2 R^2}{f} - p_\theta^2 \right)^{1/2} \times 2 \\ &= \frac{4}{\pi} \int_0^{\omega R/\sqrt{f}} dp_\theta \left(\frac{\omega^2 R^2}{f} - p_\theta^2 \right)^{1/2} = \frac{\omega^2 R^2}{f}, \end{aligned} \tag{2.8}$$

where the p_θ integration goes over those values of p_θ that make the square root positive. The free energy is given by

$$F = \frac{1}{\beta} \int dg(\omega) \ \ln(1 - e^{-\beta\omega}) = -\frac{R^2}{f} \int \frac{\omega^2 \ d\omega}{e^{\beta\omega} - 1} = -\frac{2\zeta(3) R^2}{\beta^3 f}, \tag{2.9}$$

where the zeta function $\zeta(3) = 1.202$. The entropy reads

$$S = \beta^2 \ \frac{\partial F}{\partial \beta} = \frac{6\zeta(3) R^2}{\beta^2 f}. \tag{2.10}$$

We expand the function $f(R)$ with Taylor series near the horizon

$$f(R) = f'(r_0)(R - r_0) = 2\kappa\varepsilon, \tag{2.11}$$

and $\beta^{-1} = \kappa/2\pi$, then (2.10) becomes

$$S = \frac{3\zeta(3) R^2}{2\pi\beta\varepsilon}. \tag{2.12}$$

The proper distance corresponding to ε is given by

$$\alpha = \int_{r_0}^{r_0+\varepsilon} \frac{dr}{\sqrt{f}} \approx \sqrt{\frac{2\varepsilon}{\kappa}}. \tag{2.13}$$

Then

$$S = \frac{3\zeta(3) R^2}{2\pi^2 \alpha^2} = \frac{3\zeta(3) A}{8\pi^3 \alpha^2}. \tag{2.14}$$

The B–H entropy will be obtained by regulating the value of cutoff to make the equality $\alpha^2 = \frac{3\zeta(3)}{2\pi^3}$ hold, which is different from that in the brick-wall model.

We also notice that $A = 4\pi R^2$ is not the area of the horizon but the area of the surface fixed near the horizon. However, in our opinion, A can be regarded as the "effective area" of the horizon when the fluctuations of the horizon are taken into account. To understand this viewpoint, we assume the radius of horizon has a fluctuation δ, then the mean of the area

$$\langle A\rangle = 4\pi < (r_0 + \delta)^2 > = 4\pi \left(r_0^2 + \delta^2\right) > 4\pi r_0^2, \tag{2.15}$$

That is to say, the horizon seems to have a nonzero thickness because of the quantum effects of the fields and space–time itself. In other words, the horizon may be dressed by a film of space–time foam (Scardigli, 1997). In fact, the horizon's length will increase when the back reaction to the hole is taken into account (York, 1985; Huang *et al.*, 1993). If $\langle A\rangle$ is identical with $4\pi R^2$, the cutoff is naturally provided by

$$\delta^2 = \varepsilon^2 + 2r_0\varepsilon. \tag{2.16}$$

We make further investigations of cutoff, starting with another aspect. Padmanabhan (Padmanabhan, 1999) has qualitatively pointed out that the cutoff is necessary if the locally defined energy E_{loc} of a mode is not allowed to exceed the Planck energy by an arbitrary amount. But he doesn't answer the puzzle arising from Eq. (2.12): why ε depends on the inverse temperature? We would like to discuss his viewpoint in details. In classical space–time, local energy near the horizon is given by

$$E_{loc} = \frac{\omega}{\sqrt{f}} \simeq \frac{\omega}{\sqrt{2\kappa(r - r_0)}} = \omega\sqrt{\frac{\beta}{4\pi\varepsilon}}, \tag{2.17}$$

where ω is the frequency of modes measured by the remote observer. According to Wien's displacement law, the maximum energy density of black body radiation is at the specific mode with frequency $\omega = 2.822\beta^{-1}$. If the local energy has a maximum comparable to the Planck energy,

$$E_{loc} \sim 1, \tag{2.18}$$

From (2.17) and (2.18), one can easily obtain

$$\varepsilon \sim \beta^{-1}. \tag{2.19}$$

We see that ε is indeed temperature-dependent, which makes the black hole entropy proportional to the area of the horizon. The fine computation shows that $E_{loc} \simeq 1.86$, comparable to the Planck energy. In accordance with the general concepts of the quantum theory in curved space–time, it is not allowable to ignore the fluctuations of the space–time on the Planck scale. As previously discussed, the fluctuations of the horizon influence the black hole entropy. The cutoff as a direct result of the horizon fluctuations is naturally introduced to avoid the divergence of entropy. More than 40 years ago, Pauli and other authors (cited by Ford, 1999) suggested that lightcone fluctuations could cancel the divergence of quantum field theory. Preceding discussions seem to provide an evidence for the hypothesis. The fractal spectrum of the horizon fluctuations is investigated by Sorkin (Sorkin, 1997).

3. APPLICATION TO A NON-STATIC HOLE

The merits of our model are evident. It can be applied to the system out of equilibrium, such as the Vaidya–de Sitter(V–dS) space–time that has two horizons with two correspondingly different temperatures. Furthermore, V–dS geomety is nonstatic and describes an evaporating black hole. Its temperature is time-dependent, and the horizon shrinks in the process of evaporation. We confront the difficulties of concept when we try to apply the brick-wall model to the nonstatic case, at least in the primal form based on the large scale thermodynamic equilibrium. However, our model may be applied to it. According to our model, the entropy of the hole is only attributable to the degrees of freedom of the fields covering the surface of the horizon, where the thermodynamic equilibrium is well-defined. We can investigate the properties of the membrane by using statistical mechanics. It is noted that the thermodynamic equilibrium between the hole and the external fields is not necessarily assumed, whereas this postulate is the basis for the brick-wall model.

A nonstatic space–time is described by Vaidya–de Sitter metric

$$ds^2 = -\left(1 - \frac{2m(v)}{r} - \frac{\Lambda r^2}{3}\right) dv^2 + 2\ dv\ dr + r^2\ d\Omega^2, \qquad (3.1)$$

which can be described as an evaporating black hole in de Sitter space–time. The location of event horizon r_h and radiation temperature T_H are determined by (Zhao and Dai, 1992; Li *et al.*, 1999)

$$1 - 2\ \dot{r}_h - \frac{2m}{r_h} - \frac{\Lambda r_h^2}{3} = 0, \qquad (3.2)$$

$$T_H = \frac{b}{4\pi(1 - 2\ \dot{r}_h)}, \qquad (3.3)$$

where $\dot{r}_h = \frac{dr_h}{dv}$ and $b = 2m/r_h^2 - 2\Lambda r_h/3$. The variable temperature is meaningful only in the vicinity of the horizon. The thermodynamic equilibrium is well defined on the surface of the horizon because of the spherical symmetry of the space.

One can see from (3.1) and (3.2) that the infinite red-shift surface does not coincide with the moving event horizon. We expect that there exists a frame where these two surfaces are identical. We introduce the coordinate transformation

$$r' = r - r_h, \qquad (3.4)$$

$$dr' = dr - \dot{r}_h \ dv.$$

So (3.1) can be reduced to

$$ds^2 = -\left(1 - 2\ \dot{r}_h - \frac{2m(v)}{r} - \frac{\Lambda r^2}{3}\right)\ dv^2 + 2\ dv\ dr' + r^2\ d\Omega^2, \tag{3.5}$$

where $r = r' + r_h$. An observer comoving with the event horizon is described by $dr' = 0$. The physical meaning of this coordinate transformation is easily understood. To cancel the effect caused by the motion of the horizon, we must choose a frame comoving with the horizon. Similar to the preceding computations, a spherical surface near the horizon is fixed at $r' = \epsilon$; ϵ is a small quantity and $d\epsilon = 0$. The geometry of the surface is described by

$$ds_m^2 = -\left(1 - 2\ \dot{r}_h - \frac{2m(v)}{r} - \frac{\Lambda r^2}{3}\right)\ dv^2 + r^2\ d\Omega^2, \tag{3.6}$$

And

$$\begin{aligned} &\sqrt{-g} = \sqrt{\Delta(\epsilon, v)}r^2 \sin\theta, \quad g^{00} = -\Delta^{-1}, \quad g^{11} = \frac{1}{r^2}, \\ &g^{22} = \frac{1}{r^2\ \sin^2\theta}, \end{aligned} \tag{3.7}$$

where $\Delta(\epsilon, v) = 1 - 2\ \dot{r}_h - 2m(v)/r - \Lambda r^2/3$. Substituting (3.7) into the same equation of massless scalar field as (2.3), we obtain

$$\begin{aligned} &\frac{r^2}{\Delta}\ \partial_v^2\Phi + \frac{1}{\sqrt{\Delta}}\ \partial_v\ \left(r^2/\sqrt{\Delta}\right)\ \partial_v\Phi + \partial_\theta^2\Phi + \cot\theta\ \ \partial_\theta\Phi \\ &\qquad + \frac{1}{\sin^2\theta}\ \ \partial_\varphi^2\Phi = 0. \end{aligned} \tag{3.8}$$

It is shown (Zhao and Dai, 1992) that the asymptotic behavior of the wave equation near horizon is the standard form, using the tortoise coordinate. It means that a reasonable solution of (3.8) is still assumed by WKB approximation,

$$\Phi = y(v)e^{-i\omega v + iS(\theta,\varphi)}, \tag{3.9}$$

in the nonstatic space–time. It is noted that the approximation is only valid in the vicinity of the event horizon. There is an additional term in (3.8) compared with (2.4). To cancel the difference, $y(v)$ must satisfy the following differential equation

$$\frac{r^2}{\sqrt{\Delta}}\ddot{y} + \partial_v\left(r^2/\sqrt{\Delta}\right)\dot{y} = 0, \tag{3.10}$$

where $\dot{y} = \frac{\partial y}{\partial v}$. We define $p_\theta = \frac{\partial S}{\partial \theta}$, $p_\varphi = \frac{\partial S}{\partial \varphi}$. Substituting (3.9) into (3.8), we have

$$\frac{\omega^2 r^2}{\Delta} - p_\theta^2 - \frac{1}{\sin^2\theta}p_\varphi^2 = 0, \tag{3.11}$$

or

$$p_\varphi = \pm \sin\theta \left(\frac{\omega^2 r^2}{\Delta} - p_\theta^2 \right)^{1/2}, \tag{3.12}$$

which is similar to Eq. (2.7). What we do next is similar to the preceding computations. So the entropy of a Vaidya black hole is given by

$$S = \frac{6\zeta(3) r^2}{\beta^2 \Delta}, \tag{3.13}$$

where β is the inverse temperature in (3.3). We expand the Δ, with Taylor series, near the horizon:

$$\Delta = 1 - 2\,\dot{r}_h - \frac{2m}{r_h + \epsilon} - \frac{\Lambda r^2}{3} \simeq \left(\frac{2m}{r_h^2} - \frac{2\Lambda r_h}{3} \right) \epsilon = b\epsilon. \tag{3.14}$$

Then (3.13) becomes

$$S = \frac{6\zeta(3) r^2}{\beta^2 b \epsilon}, \tag{3.15}$$

A new cutoff is redefined as

$$\alpha' = \int_0^{\epsilon'} \frac{dr'}{(1 - 2\,\dot{r}_h - 2m/r - \Lambda r^2/3)^{1/2}} \simeq 2\sqrt{\frac{\epsilon'}{b}}, \tag{3.16}$$

where $\epsilon' = \epsilon(1 - 2\,\dot{r}_h)^2$, and

$$\frac{4\epsilon'}{b} = \alpha'^2, \tag{3.17}$$

Substituing (3.3) and (3.17) into (3.13), we have

$$S = \frac{3\zeta(3) r^2}{2\pi^2 \alpha'^2} = \frac{3\zeta(3) A'}{8\pi^3 \alpha'^2}, \tag{3.18}$$

where $A' = 4\pi r^2 \simeq 4\pi r_h^2$. The cutoff α' is time-dependent, since ϵ is defined as a constant. It may be a natural choice in nonstatic space–times.

In summary, we propose a model to compute the statistical entropy of the black hole. In our opinion, the degrees of the fields covering the 2-D surface just outside the horizon are responsible for the black hole entropy. A cutoff is necessary to avoid the divergence of the entropy. We propose that it is not allowable to ignore the fluctuations of the horizon because the locally defined energy of modes close to the horizon may exceed the Planck energy. Therefore, the cutoff introduced can be treated as the increase in horizon for the quantum fluctuations of space–time. In other words, the horizon has a thickness of the Planck length. The thickness is so small that we are not able to investigate its properties by using the quantum theory in curved space. What we can only do is to know indirectly the statistical properties of the membrane by studying the fields propagating on its surface. By using the membrane approach, the statistical entropy of Vaidya–de Sitter hole is calculated and still proportional to the area of the horizon. The result relies on a time-dependent cutoff.

An idea similar to the membrane approach is abstractly expressed (Calip and Teitelboim, 1995; Calip, 1995), where the thermodynamics of the 2+1 black hole (not the 3+1 case!) is discussed using Chern–Simons theory. Compared to it, the picture of our method is simple but clear. A crucial difference is that our model can easily be applied to the nonstatic hole.

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